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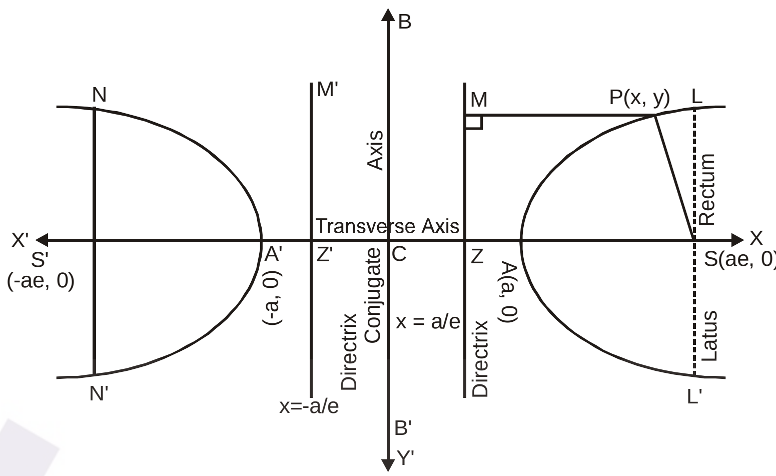
HYPERBOLA

EQUATION OF HYPERBOLA IN STANDARD FORM

The general form of standard hyperbola is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where a and b are constants.



TERMS RELATED TO A HYPERBOLA

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, has been shown in the figure given above.

Symmetry Since only even powers of x and y occur in the above equation, so the curve is symmetrical about both the axes.

Foci If S and S' are the two foci of the hyperbola and their coordinatesd are (ae, 0) and (–ae, 0) respectively, then distance between foci is given by SS' = 2ae.

Directries ZM and Z' M' are the two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance directrices is given by zz' = $\frac{2a}{e}$.

Axes The lines AA' and BB' are called the transverse axis and conjugate axis respectively of the hyperbola.

The length of transverse axis = AA' = 2a

The length of conjugate axis = BB' = 2b

Centre The point of intersection C of the axes of hyperbola is called the centre of the hyperbola. All chords, passing through C, are bisected at C.

Vertices The points A $\equiv$ (a, 0) and A' $\equiv$ (–a, 0) where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.

Focal Chord A chord of the hyperbola passing through its focus is called a focal chord.

Focal Distances of a Point The difference of the focal distances of any point on the hyperbola is constant and equal to the length of the transverse axis of the hyperbola. If P is any point on the hyperbola, then

S'P – SP = 2a = Transverse axis.

Latus Rectum If LL' and NN' are the latus rectum of the hyperbola then these lines are perpendicular to the transverse axis AA', passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \qquad L' \equiv \left(ae, \frac{-b^2}{a}\right),$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), \qquad N' \equiv \left(-ae, \frac{-b^2}{a}\right).$$

Length of latus rectum = LL' = $\frac{2b^2}{a}$ = NN'.

Eccentricity of the Hyperbola We know that

$$SP = e\,PM \quad \text{or} \quad SP^2 = e^2\,PM^2$$

or $$(x - ae)^2 + (y - 0)^2 = e^2\ N' \equiv \left(x - \frac{a}{e}\right)^2$$

$$(x - ae)^2 + y^2 = (ex - a)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 - 2aex + a^2$$

$$x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1)$$

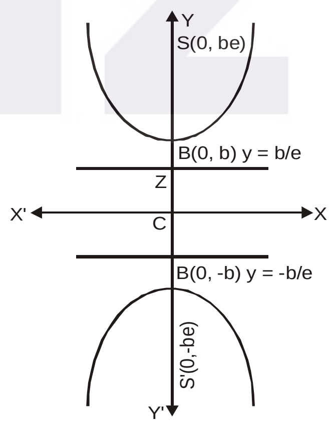
$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1.$$

On comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $b^2 = a^2 (e^2 - 1)$ or $e = \sqrt{1 + \frac{b^2}{a^2}}$

PARAMETRIC EQUATIONS OF THE HYPERBOLA

Since coordinates $x = a \sec\theta$ and $y = b \tan\theta$ satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

for all real values of q therefore, $x = a \sec\theta$, $y = b \tan\theta$ are the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.



Hence, the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as $(a \sec\theta, b \tan\theta)$. This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point $(a \sec\theta, b \tan\theta)$ on the hyperbola.

Equation of Chord The equation of the chord joining the points

$$P \equiv (a \sec\theta_1, b \tan\theta_1) \text{ and } Q \equiv (a \sec\theta_2, b \tan\theta_2) \text{ is}$$

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right) \quad \text{or} \quad \begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & b\tan\theta_1 & 1 \\ a\sec\theta_2 & b\tan\theta_2 & 1 \end{vmatrix}=0$$

CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } \qquad -\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \left(\text{i.e., } \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1\right)$$

PROPERTIES OF HYPERBOLA AND ITS CONJUGATE

	Hyperbola	Conjugate Hyperbola
Standard equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{-x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Equation of transverse axis	y = 0	x = 0
Equation of conjugate axis	x = 0	y = 0
Length of transverse axis	2a	2b
Length of Conjugate axis	2b	2a
Foci	(± ae, 0)	(0, ± be)
Equation of directrices	x = ± a/e	y = ± b/e
Vertices	(± a, 0)	(0, ± b)
Eccentricity	$e=\sqrt{\frac{a^2+b^2}{a^2}}$	$e=\sqrt{\frac{a^2+b^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parameter Coordinates	$(a\sec\theta, b\tan\theta)$	$(b\sec\theta, a\tan\theta)$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii (S'P – SP)	2a	2b
Tangent at the vertices	x = ± a	y = ± b

POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

The point $P(x_1, y_1)$ lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0.$ $=0$ or <0.

CONDITION FOR TRANGENCY AND POINTS OF CONTACT

The condition for the line y = mx + c to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

EQUATION OF TANGENT IN DIFFERENT FORMS

Point Form The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Note : The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y_2 by yy_1, x by $\frac{x+x_1}{2}$,y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of hyperbola is a polynomial of second degree in x and y.

Parametric Form The eq^n of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (a secθ, b tanθ) is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

Slope Form The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm \sqrt{a^2m^2-b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

Notes :

· Number of Tangents From a Point Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the hyperbola.

· Director Circle It is the locus of points from which $\perp$ tangents are drawn to the hyperbola. The equation of director circle of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

EQUATION OF NORMAL IN DIFFERENT FORMS

Point Form The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

Parametric Form The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (a secθ, b tanθ) is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

Slope Form The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

Notes :

The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$$

· Number of Normals

In general, four normals can be drawn to a hyperbola from a point in its plane i.e., there are four points on the hyperbola, the normals at which will pass through a given point. These four points are called the co-normal points.

· Tangent drawn at any point bisects the angle between the lines joining the point to the foci, whereas normal bisects the supplementary angle between the lines.

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point

$P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$SS_1 = T^2$

where $S \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1$, $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

and $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

CHORD WITH A GIVEN MID POINT

The equation of chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by $T = S_1$ where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$ and $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T = 0$, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

POLE AND POLAR

The polar of a point $P(x_1, y_1)$ w.r.t. the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T = 0$, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

Notes :

· Pole of a given line $lx + my + n = 0$ w.r.t. the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } \left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$$

- Polar of the focus is the directrix.
- Any tangent is the polar of its point of contact.
- If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to the conjugate points.
- If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

EQUATION OF A DIAMETER OF A HYPERBOLA

The equation of the diameter bisecting chords of slope m of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2 m}$.

CONJUGATE DIAMETERS

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chord parallel to the other. If m_1 and m_2 be the slopes of the conjugate diameters of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m_1 m_2 = \frac{b^2}{a^2}$

ASYMPTOTES OF HYPERBOLA

The lines $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ i.e., $y = \pm \frac{bx}{a}$ are called the asymptotes of the hyperbola.

The curve comes close to these lines as $x \to \infty$ or $x \to -\infty$ but never meets them. In other words, asymtote to a curve touches the curve at infinity.

Note :

- The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
- Asymptotes are the diagonals of the rectangle passing through A, B, A', B' with sides parallel to axes.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The asymptotes pass through the centre of the hyperbola.
- The bisector of the angle between the asymptotes are the coordinates axes.
- The product of the perpendicular from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is a constant equal to $\frac{a^2 b^2}{a^2 + b^2}$.
- Any line drawn parallel to the asymptote of the hyperbola would meet the curve only at one point.
- A hyperbola and its conjugate hyperbola have the same asymptotes.

RECTANGULAR HYPERBOLA

If asymptotes of the standard hyperbola are perpendicular to each other, then it is known as Rectangular Hyperbola. Then

$$2\tan^{-1}\frac{b}{a} = \frac{\pi}{2} \Rightarrow b = a \text{ or } x^2 - y^2 = a^2$$

is general form of the equation of the rectangular hyperbola.

If we take the coordinate axes along the asymptotes of a rectangle hyperbola, then equation of rectangular hyperbola becomes : $xy = c^2$, where c is any constat.

In parametric form, the equation of rectangular hyperbola

$x = ct$, $y = c/t$, where t is the parameter.

The point (ct, c/t) on the hyperbola $xy = c^2$ is generally referred as the point 't'.

Properties of Rectangular Hyperbola, $x^2 - y^2 = t^2$

· The equations of asymptotes of the rectangular hyperbola are $y = \pm x$.

· The transverse and conjugate axes of a rectangular hyperbola are equal in length.

· Eccentricity, $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{2}$.

Properties of Rectangular hyperbola $xy = c^2$

· Equation of the chord joining 't_1' and 't_2' is

$$x + yt_1t_2 - c(t_1 + t_2) = 0$$

· Equation of tangnet at (x_1, y_1) is

$$xy_1 + x_1y = 2c^2 \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2$$

· Equation of tangent at 't' is : $\frac{x}{t} + yt = 2c$.

· Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1t_2}\right)$

· Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

· Equation of normal at 't' is: $xt^3 - yt - ct^4 + c = 0$

· The equation of the chord of the hyperbola $xy = c^2$ whose middle point is (x_1, y_1) is $T = S_1$ i.e., $xy_1 + x_1y = 2x_1y_1$.

· The slope of the tangent at the point (ct, c/t) is $-1/t^2$, which is always negative. Hence tangents drawn at any point to $xy = c^2$ would always make an obtuse angle with the x-axis.

· The slope of the normal at the point (ct, c/t) is t^2 which is always positive. Hence normal drawn to $xy = c^2$ at any point would always make an acute angle with the x-axis.